The Road to Chaos

Discrete Self-Similarity and the Golden Ratio in Bianchi-IX Cosmologies

David W. Hobill

Department of Physics and Astronomy

University of Calgary

BIANCHI - IX COSMOLOGIES: METRIC METHOD

1. Metric has THREE scale factors a, b, c, in three orthogonal directions.

$$\begin{aligned} ds^2 &= -dt^2 + c^2(t)d\psi^2 + \left[a^2(t)\cos^2\psi + b^2(t)\sin^2\psi\right]d\theta^2 \\ &+ \left[a^2(t) - b^2(t)\right]\sin 2\psi \sin\theta d\theta d\phi + 2c^2(t)\cos\theta d\psi d\phi \\ &+ \left[\left(a^2(t)\sin^2\psi + b^2(t)\cos^2\psi\right)\sin^2\theta + c^2(t)\cos^2\theta\right]d\phi^2. \end{aligned}$$

2. Einstein Equations reduce to ODE's

$$\frac{d}{dt}\left(\frac{da}{dt}bc\right) = \frac{1}{2abc}\left[\left(b^2 - c^2\right)^2 - a^4 + (2-\gamma)\rho a^2b^2c^2\right],$$

$$\frac{d}{dt}\left(a\frac{db}{dt}c\right) = \frac{1}{2abc}\left[\left(c^2 - a^2\right)^2 - b^4 + (2-\gamma)\rho a^2b^2c^2\right],$$

$$\frac{d}{dt}\left(ab\frac{dc}{dt}\right) = \frac{1}{2abc}\left[\left(a^2 - b^2\right)^2 - c^4 + (2-\gamma)\rho a^2b^2c^2\right],$$

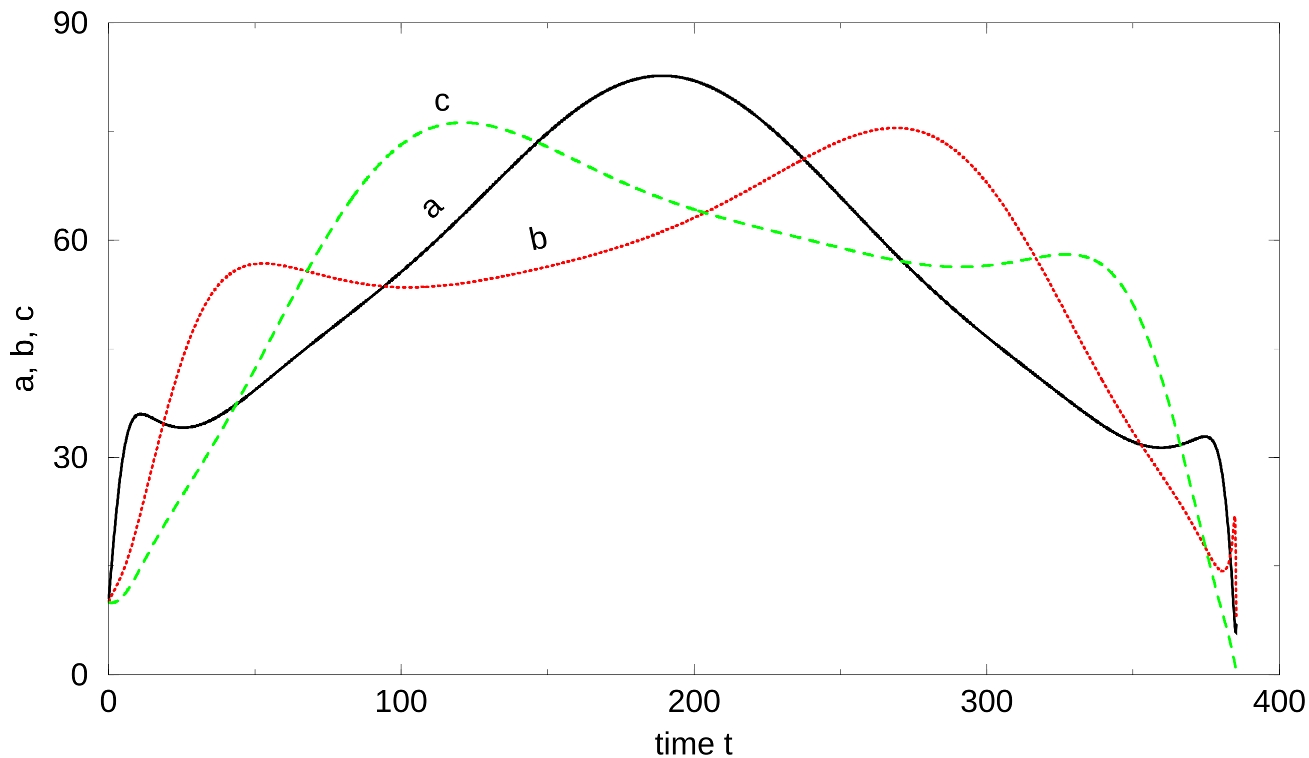
Subject to the constraint:

$$\frac{d}{dt}\left(\frac{da}{dt}bc + a\frac{db}{dt}c + ab\frac{dc}{dt}\right) - 2\left(\frac{da}{dt}\frac{db}{dt}c + \frac{da}{dt}b\frac{dc}{dt} + a\frac{db}{dt}\frac{dc}{dt}\right) = -\frac{1}{2}(3\gamma - 2)\rho abc$$

EOS: $p = (\gamma - 1)\rho$, [$\gamma = 1$, dust; $\gamma = \frac{4}{3}$, radiation; $\gamma = 0$, Cosmological const.]

B-IX EVOLUTION: ENTIRE COSMOLOGICAL HISTORY

Bianchi-IX closed cosmological models with oscillating scale factors
Oscillations produce accelerations and decelerations:



Behaviour similar to a quintessence driven evolution.

BIANCHI - IX COSMOLOGIES: FRAME METHOD

Spatial-Curvature (N_i) and Shear Variables ($\Sigma_\pm$) (Ellis-MacCallum-Wainwright)

$$\begin{aligned}
\dot{N}_1 &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_1, \\
\dot{N}_2 &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_2, \\
\dot{N}_3 &= (q - 4\Sigma_+)N_3, \\
\dot{\Sigma}_\pm &= -(2-q)\Sigma_\pm - S_\pm,
\end{aligned}$$

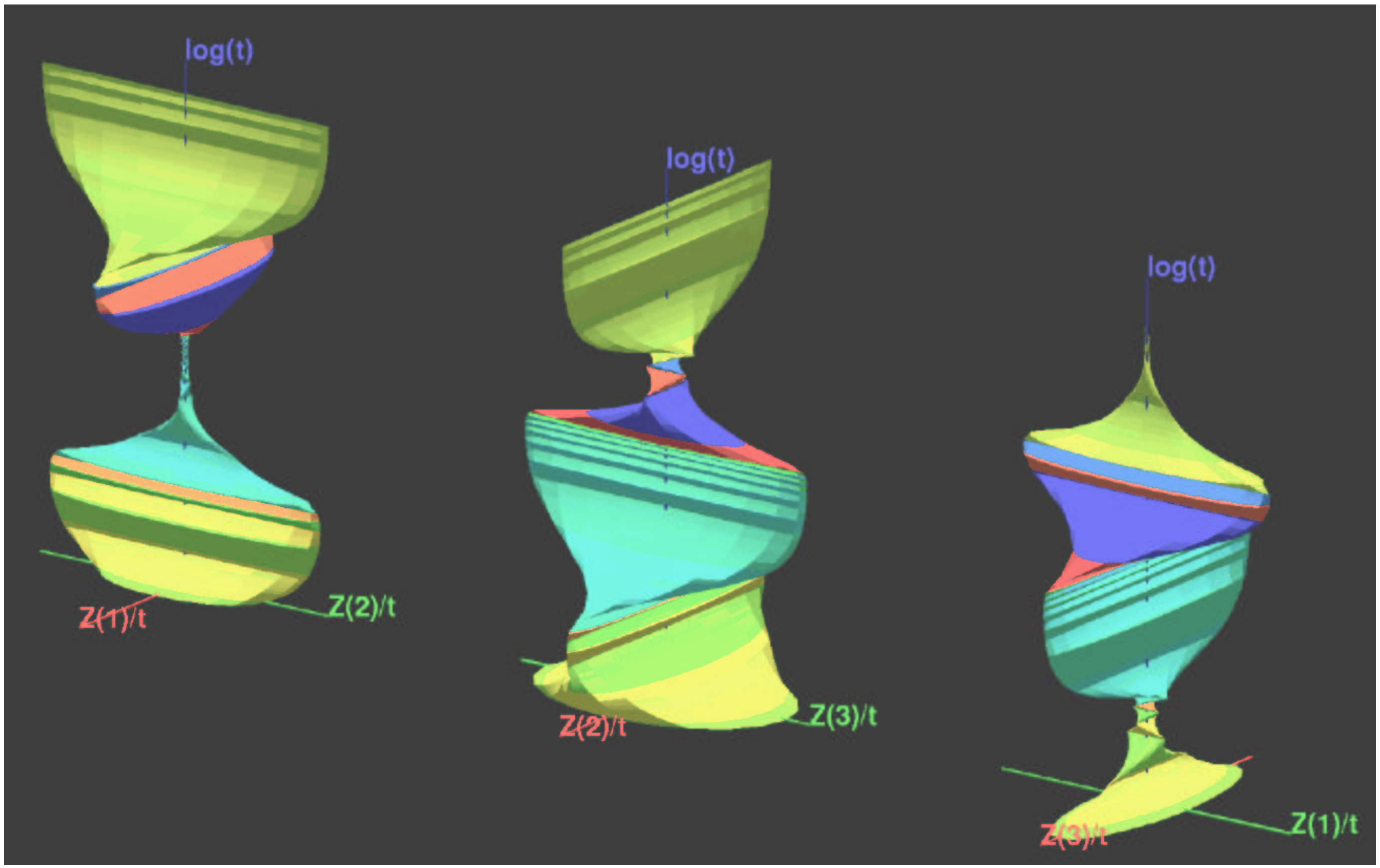
where $\dot{\{\}} = d/d\tau$ $(d\tau = dt/H)$ and:

$$\begin{aligned}
q &= \frac{1}{2}\left[(3\gamma - 2)(1-K) + 3(2-\gamma)(\Sigma_+^2 + \Sigma_-^2)\right], \\
K &= \frac{1}{12}\left[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_1N_3)\right], \\
S_+ &= \frac{1}{6}\left[(N_1 - N_2)^2 - N_3(2N_3 - N_1 - N_2)\right], \\
S_- &= \frac{1}{2\sqrt{3}}(N_2 - N_1)(N_3 - N_1 - N_2),
\end{aligned}$$

$$p = (\gamma - 1)\rho$$

B-IX EVOLUTION: EARLY UNIVERSE

Solutions constructed using numerical Runge-Kutta methods
Nonlinear dynamical system: Sensitive Dependence on Initial Conditions (Chaos)



Here $Z_i = \ln N_i$

Generic Solution: Sequence of Eras and Epochs

Special Solutions in Frame Variables

Solution for Friedman-Lemaitre-Robertson-Waker isotropic spacetime:

$$\Sigma_\pm = 0 \quad \text{and} \quad N_1 = N_2 = N_3 = K_F{}^{LRW}$$

Solution for the Kasner (Bianchi I) spacetime:

$$\Sigma_+^2 + \Sigma_-^2 = 1 \quad \text{and} \quad N_1 = N_2 = N_3 = 0$$

Solution for the Taub (Bianchi II) spacetime:

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}N_i^2 = 1, \quad \text{and} \quad N_j = N_k = 0$$

$$\{i, j, k\} = \text{permutation}\{1, 2, 3\}$$

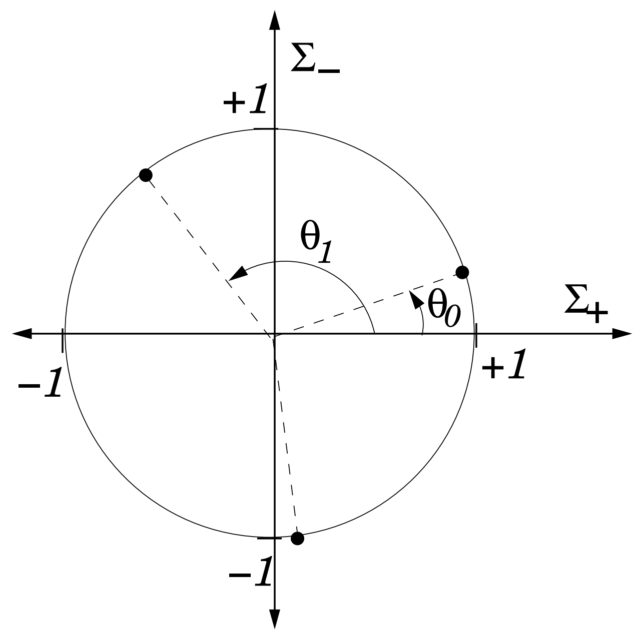
Near the singularity Bianchi-IX solutions converge to Kasner-like epochs separated by Taub solutions.
Kasner states define the unit circle in the shear plane: $\Sigma_+^2 + \Sigma_-^2 = 1$

Taub states define a 3D ellipsoid $\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}N_i^2 = 1$

BOGOYAVLENSKY MAP

B-Map follows transitions from one Kasner epoch to another (Kasner ring transitions)

1D Map on Kasner Ring of the form: $\theta_{(n+1)} = F(\theta_n)$

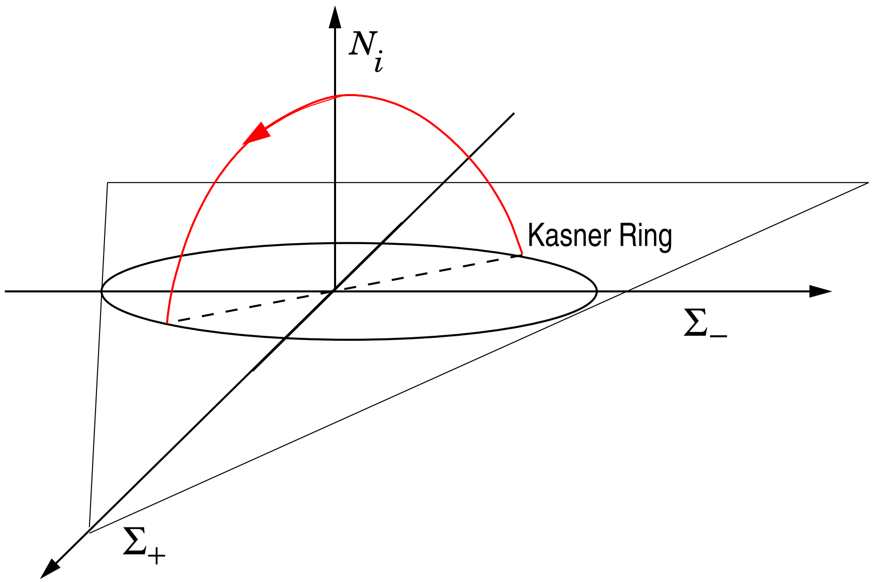


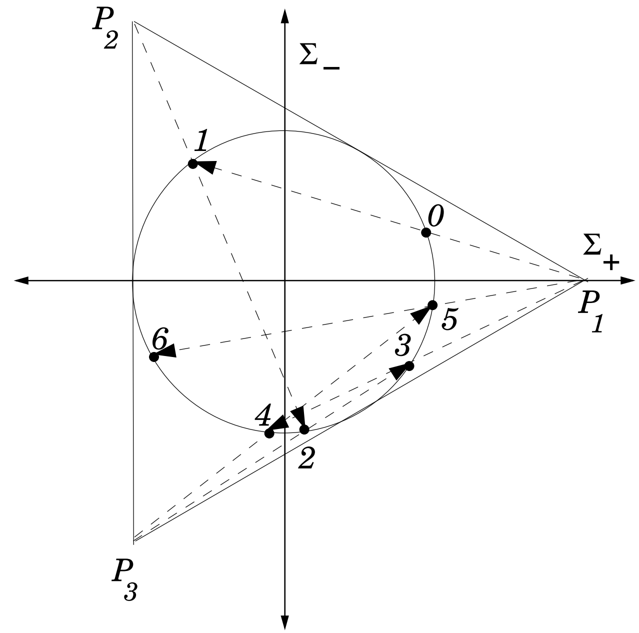
$$\theta_{(n+1)} = \cos^{-1}\left(\frac{4 - 5\cos\theta_n}{5 - 4\cos\theta_n}\right)$$

Originally obtained using BKL (metric) variables

GEOMETRIC CONSTRUCTION - (Wainwright-Ma)

Circumscribe equilateral triangle around Kasner ring in shear plane





Each crossing of Kasner circle described by a Taub vacuum solution
Transition occurs on surface of the Taub ellipsoid

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4} N_i^2 = 1$$

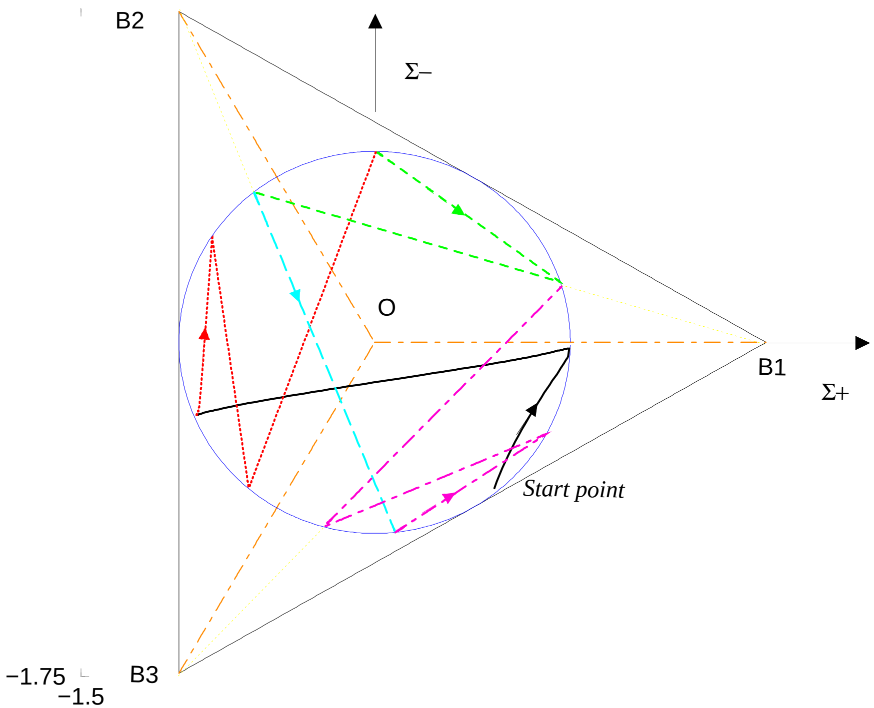
only one N_i nonzero - others vanish during transition

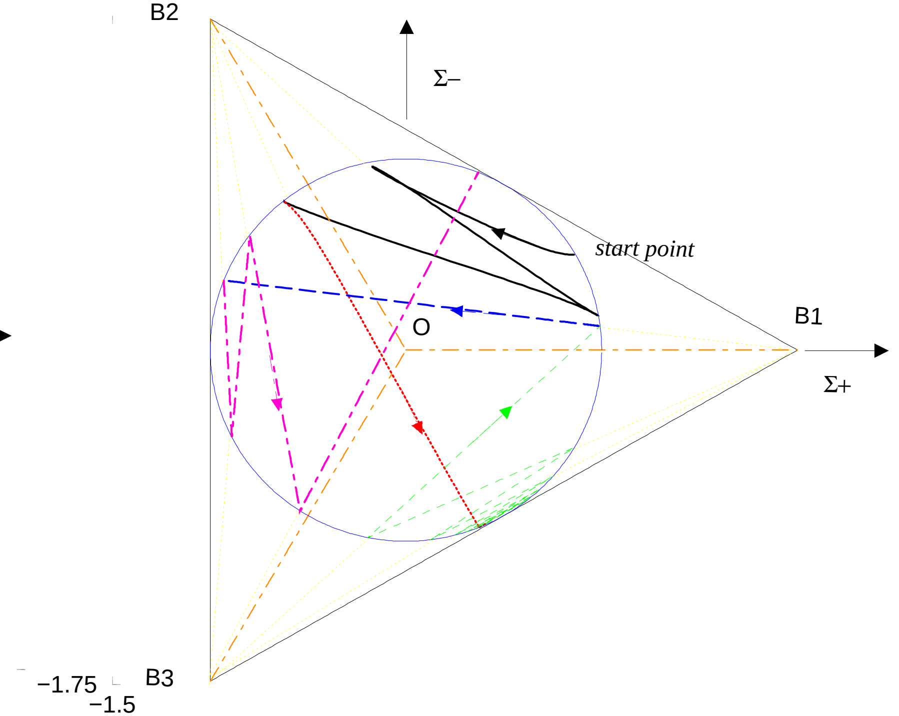
Projection on Shear Plane defines B-Map

Exact Dynamics Projected onto Shear Plane

Solve ODE's using numerical Runge-Kutta methods

Map Σ_+ andΣ_- onto the shear plane

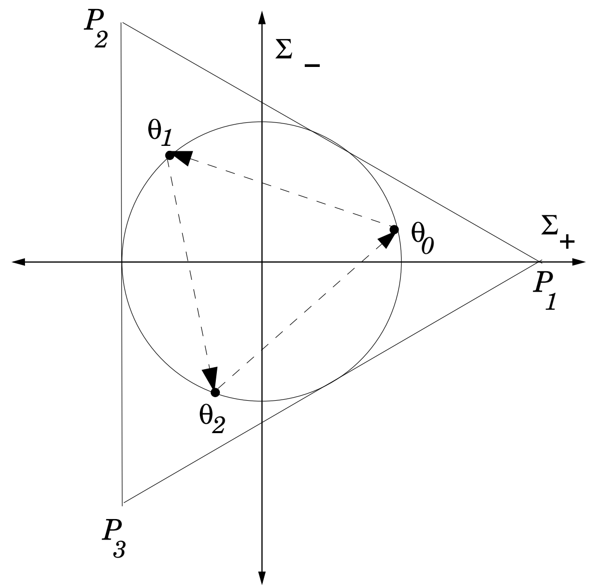




Black curves - transient behaviour from initial data

Late time behaviour converges to the Kasner Map solutions

A Periodic Solution of Kasner Map



Kasner Map repeats after three transitions

"Period-3 Implies Chaos"[a]

Equivalent Bogoyavlensky map

$$\theta_{(n+1)} = \theta_n + \frac{2\pi}{3} = \cos^{-1}\left(\frac{4 - 5\cos\theta_n}{5 - 4\cos\theta_n}\right)$$

leads to a quartic equation for $\cos(\theta_n)$

$$64\cos^4\theta - 80\cos^3\theta - 12\cos^2\theta + 40\cos\theta - 11 = 0$$

---

[a] Li, T-Y & Yorke, J.A., *Amer. Math. Monthly*, **10**, 985, (1970).

Locating the Kasner Solutions in 3-cycle

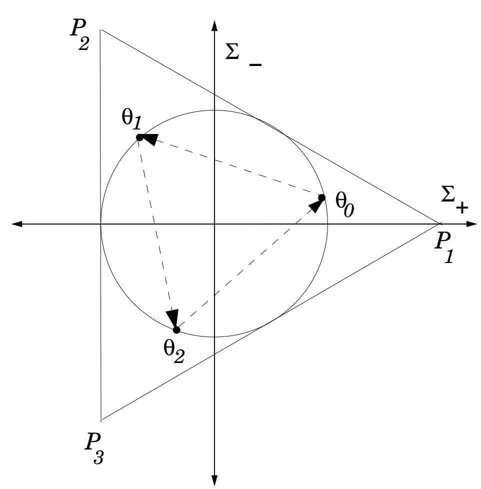
Period 3-solutions (Real Solutions to quartic) on the Kasner Ring are given by:

$$(\Sigma_+, \Sigma_-)_n = (\cos\theta_n, \sin\theta_n)$$

$$(\Sigma_+, \Sigma_-)_0 = \left(\frac{1+3\sqrt{5}}{8}, \frac{\sqrt{3}(\sqrt{5}-1)}{8}\right)$$

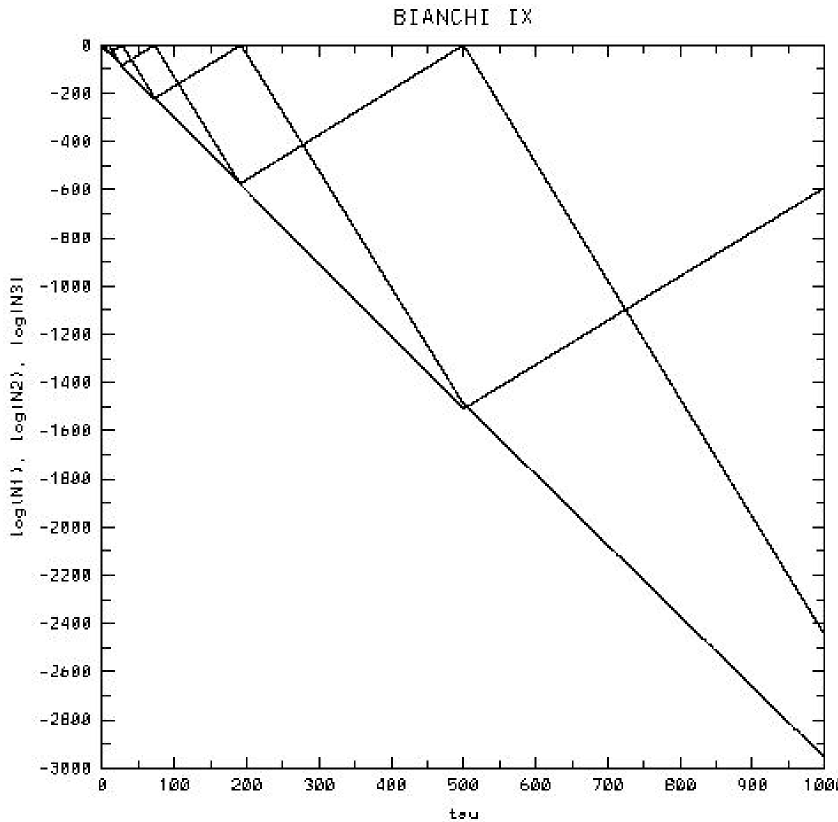
$$(\Sigma_+, \Sigma_-)_1 = \left(\frac{1-3\sqrt{5}}{8}, \frac{\sqrt{3}(\sqrt{5}+1)}{8}\right)$$

$$(\Sigma_+, \Sigma_-)_2 = \left(-\frac{1}{4}, -\frac{\sqrt{3}\sqrt{5}}{4}\right)$$



Full dynamics of 3-cycles for Curvature Variables

$(\Sigma_+, \Sigma_-)_0$ as input to initial values for full ODE evolution



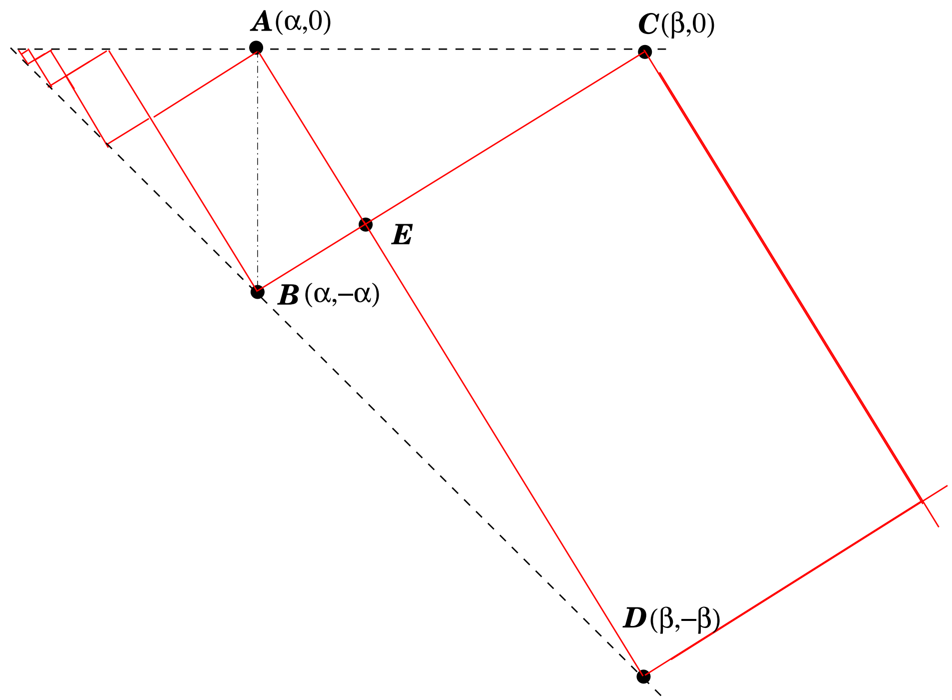
Here define $N_i = \exp(Z_i)$
Lowest Frequency Oscillations generate self-similar “Golden Rectangles”???

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.6180339887 \cdots$$

Golden Rectangle Construction

Assumptions

1. Highest and lowest vertices of each rectangle have same x coordinate
2. Lowest vertex touches $y = -x$
3. Highest vertex touches $y = 0$
4. Adjacent rectangles touch vertices where longest sides intersect shortest sides at right angles.



Result:

A. Slopes of short sides = $1/\Phi$, Slopes of long sides = -Φ

B. The rectangles are Golden Rectangles (aspect ratio = 1:Φ)

C. The sides of the rectangles scale as $\Phi^2 = 1 + \Phi$.

Einstein Equations and Justification of Assumptions

Condition 1: Kasner transitions occur simultaneously (single parameter)
(i) Compute the slopes of $\log(N_i) = Z_i$ for piece-wise linear behaviour

$$\begin{aligned} \dot{Z}_1 &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-), \\ \dot{Z}_2 &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-), \\ \dot{Z}_3 &= (q - 4\Sigma_+), \end{aligned}$$

Asymptotic value for q ($N_i \to 0$) is $q \to 2$ (independent of γ).
Equation of state not important near the cosmological singularity! (BKL)

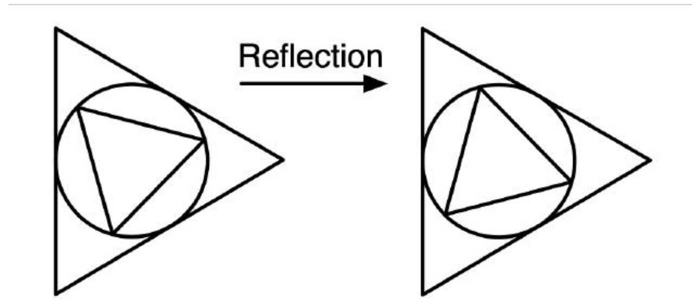
	θ_0	θ_1	θ_2
$\dot{Z}_1$	$3\Phi^{-1}$	-3Φ	$-3(1)$
$\dot{Z}_2$	-3Φ	$-3(1)$	$3\Phi^{-1}$
$\dot{Z}_3$	$-3(1)$	$3\Phi^{-1}$	-3Φ

After rescaling by factor of 3 (re-scale time τ by expansion $\Theta = 3H$)
Condition 2: Bounding slope of transitions is -1 ($y = -x$)
Condition 4: Intersections are perpendicular [$(-\Phi) \times (1/\Phi) = -1$]
(ii) Find $N_{\max}$ using Taub transitions

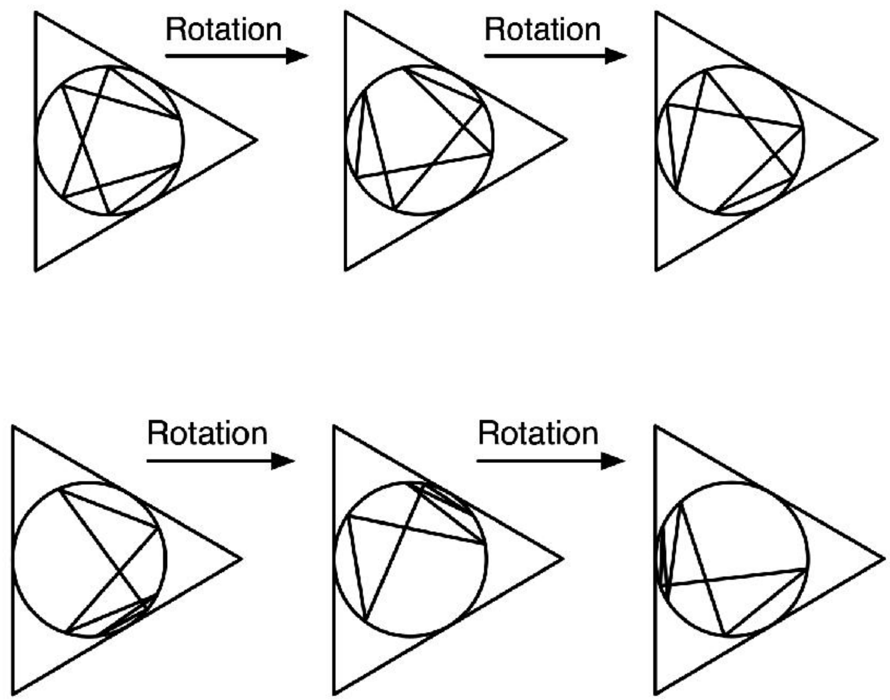
$$\Sigma_+ = \frac{1}{8}, \quad \Sigma_- = \frac{\sqrt{15}}{8} \quad \Rightarrow \quad (\Sigma_+^2 + \Sigma_-^2)_{\min} = \frac{1}{4}$$

Therefore $N_{\max} = 1$ or equivalently $Z_{\max} = 0$ (Condition 3: upper bound $y = 0$)

Other Periodic cycles of the Kasner Map



Period 3



Period 6

Open Questions

1. What structures do other periodic solutions of B-Map represent?
2. Are succesive periodic solutions related to any known sequences and/or series?
3. What structure does period-3 produce in BKL approach?
4. Are perturbations of period-3 the “most chaotic” solution of B-IX?
5. Does the B-IX period-3 solution represent a new geometric construction for Φ

ACKNOWLEDGEMENTS

University of Calgary Students: Scott Macdonald, Teviet Creighton, Mark Bryant

U Georgia Oceanographer: Adrian Burd