The Classification of the Nonsymetrical, Relativistic Models, with Applications in Astrophysics and Cosmology.

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1. Introduction

Let us consider an n-dimensional, C^{∞} differentiable manifold M_n which is a paracompact and connected one.

In such conditions , globally , there exist riemannian metrics $\{g\}$ and linear connections $\{D\}$, on M, but also linear connections having the torsion $T = 0$ (i.e. symmetrical ones).

There exists a linear connection ∇ which is compatible with g $(\nabla g = 0)$ but also is symmetrical one $\left(\overset{\nabla}{T} = 0\right)$, the well-known Levi-Civitta connection.

In cosmological theories or in the Einstein's theory of the generalised relativity g is a pseudo-riemmanian metric. Its existence supplementary requires few topological conditions , such as the vanishing of the invariant given by the Euler-Poincaré characteristic , which is equivalent with the existence of a vector field $X \in \mathscr{X}(M)$ such that $X \neq 0$ in every point .

The model of the Einstein Generalised Relativity is based on the Levi-Civitta connection ∇, which will be denoted by $L_{\nabla} = \{M; g; \nabla\}$.

If we consider a model of a physical or cosmological phenomenon with another connection D we will denote it by $L_D = (M, g, D)$.

Obviously we have to study if we can consider those models of the Einstein Generalised Relativity with $D \neq \nabla$.

Besides the well-known notices regarding the model $L_D = (M, g, \nabla)$, at the begining of his research Einstein has asked himself:

a) What kind of meaning can have the models $L_D = (M, g, D)(D \neq \nabla)$?

b) Does the Einstein 's equations have only a formal meaning in D?

For a long time he has not been agreeing these models.

Nevertheless he gets to the conclusion that , in certain conditions , such kind of models $L_D = (M; g, D)$ can be also considered. (See „The Meaning of Relativity", Fifth edition including „The Relativistic Theory of the Non-Symetric Field „([7]). But he has chosen the linear connection D such that , even $T \neq 0$, through certain transformations he will have get also to the symmetrical case $L_\nabla = (M; g; \nabla)$, with the symmetrical energy-momentum tensor verifying the conservation law.

This statement suggested us the idea to see which is , at the core , the central point of the link between the nonsymmetrical model and the symmetrical one , with the aim of generalising this model. Otherwise which is the most general link between these two models L_∇ and L_D?

In this way we can obtain classes of relativistic models , even with another cosmological interpretations.

§1.Non-symmetrical equivalent models

Let us consider two geometrical models $L_{\overset{(1)}{D}} = \left(M; g, \overset{(1)}{D}; F \right); L_{\overset{(2)}{D}} = \left(M; g, \overset{(2)}{D}; F \right)$ for the same physical or cosmological phenomena $\{F\}$, with the two linears connections $\overset{(1)}{D}, \overset{(2)}{D}$ which are not compulsory symmetrical ones.

Let us consider two one-dimensional arbitrary distributions $\overset{(1)}{\mathcal{D}}_1, \overset{(2)}{\mathcal{D}}_1$ which are orthogonal ones :

(1.1) $$g(X, Y) = 0; X \in \overset{(1)}{\mathcal{D}}_1, Y \in \overset{(2)}{\mathcal{D}}_1$$

with the well-known meaning from the differential geometry.

Definition 1. If the orthogonality of these two distributions is preserved at the parallel transport related to $\overset{(1)}{D}, \overset{(2)}{D}$ then we will say that $L_{\overset{(1)}{D}}, L_{\overset{(2)}{D}}$ are two equivalent models and we will denote :

(1.2) $$L_{\overset{(1)}{D}} \overset{g}{\sim} L_{\overset{(2)}{D}}$$

The above condition is a very general one and it is very useful for our models study.

It was already established ([2], [4]) :

Theorem (1.1). We have $L_{\overset{(1)}{D}} \overset{g}{\sim} L_{\overset{(2)}{D}}$ if and only if

(1.3) $$\alpha(X) g(Y, Z) = \left(\overset{(1)}{D}_X \, g \right)(YZ) - g\left(Y, \overset{(12)}{\tau}(X, Z) \right)$$

$X, Y \in \mathfrak{X}; \alpha \in \Lambda_1(M)$, where

$$\overset{(12)}{\tau}(X,Z)=\overset{(2)}{D}_X\,Z-\overset{(1)}{D}_X\,Z$$

From (1.3) it results :

Theorem (1.2). We have $L_{\overset{(2)}{D}}\overset{g}{\sim}L_{\overset{(1)}{D}}$ if and only if

(1.4) $$\alpha(X)g(YZ)=\left(\overset{(2)}{D}_X\,g\right)(YZ)+g\left(Y,\overset{(12)}{\tau}(X,Z)\right)$$

From these theorems we get a set of important properties for our modelling studies . One of the most important is given by :

Theorem (1.3). If $\overset{(1)}{D}g=0;\overset{(2)}{D}g=0$ and $\overset{1}{D}\neq\overset{(2)}{D}$ then the two models $L_{\overset{(1)}{D}},L_{\overset{(2)}{D}}$ can not be g-equivalent ones.

Proof. From (1.3) and (1.4) if

(1.5) $$\overset{(1)}{D}g=0,\overset{(2)}{D}g=0$$

it results

(1.7) $$\overset{(12)}{\tau}(X,Z)=0 \qquad \forall X,Z\in\mathfrak{X}(M)$$

or $\overset{(1)}{D}=\overset{(2)}{D}$

§2. The class of the nonsymmetrical models which are equivalents with the symmetrical Einstein model

Let us consider a non-symmetrical Einstein model $L_D=(M,g,D)$.

It results:

Theorem (2.1). If $L_D \overset{g}{\sim} L_\nabla$ then D can not be a g-compatible linear connection.

Proof. We get the conclusion from the Theorem (1.3).

Namely for no one nonsymmetrical model L_D from the Generalised Relativity which is g-echivalent with L_∇, the linear connection D is not a g-compatible one. Therefore we have the set $\{D / Dg \neq 0\}$.

It remains only the question if there exists models L_D with $\overset{D}{T} = 0, Dg \neq 0$ which are g-echivalent with L_∇.

From (1.3) with $\overset{1}{D} = \nabla$ and $\overset{(2)}{D} = D$ because $\nabla g = 0$ and $\overset{\nabla}{T} = 0$ it results

(2.1) $\alpha(X) g(Y, Z) - \alpha(Z) g(Y, X) = 0$

Therefore :

(2.2) $g(Y, \alpha(X)Z) = g(Y, \alpha(Z)X) = 0$

Or , because X, Z are arbitrary vector fields it results $\alpha = 0$. Namely $D = \nabla$ which is not true .

So there not exists a model $L_D = (M, g, D)$ with $\overset{D}{T} = 0$ which is g-echivalent with L_∇.

It results :

Theorema (2.2). There is no model $L_D = (M, g, D)$ with $Dg = 0$ or with $\overset{D}{T} = 0$ which is g-equivalent with L_∇.

Theorem (2.3). If $L_D = (M, g, D) \overset{g}{\sim} L_\nabla$ then the linear connection D is a semi-symmetrical one , but it is not from a K. Yano-type .

Proof. If $L_D \overset{g}{\sim} L_\nabla$ it results:

(2.3) $D_X Z = \nabla_X Z - \alpha(X)_\bullet Z$

(2.4) $(D_X g)(YZ) = 2\alpha(X) g(Y,Z), \quad \alpha \neq 0$

(2.5) $T(XZ) = D_X Z - D_Y Z - [XZ] =$

$= -\alpha(X)Z + \alpha(Z)X \qquad \forall X, Z$

Namely D is a coparallel with ∇ linear connection (which is a symmetrical one) , so D is a semi-simetrical one , but it is not a g-compatible one. Therefore D is not from a K. Yano type.

The nonsymmetrical Einstein model is given by :

(2.6) $D_X Z = \nabla_X Z + \beta(X)_\bullet Z \quad \beta \in \Lambda_1(M)$

where β is an 1-exact form

(2.7) $\beta = df \qquad f \in \mathcal{F}(M)$

It is already known that every 1-exact form is a closed one , i.e.:

(2.8) $d\beta = 0$ where d is the external differential , but , according to the Poincaré ' s Lemma, the converse is only locally true.

It results :

Theorem (2.4). The nonsymmetrical Einstein model L_D is g-equivalent with the symmetrical Einstein model L_∇ and we have :

(2.9) $D_X g = -2\beta(X) g, \ \beta \neq 0$

where β is an 1-exact form and D is a semi-symmetrical linear connection which is not from K. Yano-type.

From the above theorems it results :

Theorem (2.5). A nonsymmetrical relativistic model $L_D = (M, g, D)$ which is g-equivalent with the model L_∇ is more general than the nonsymmetrical Einstein model L_{ED} .

Obviously we have the next problems :

(1) Where is involved the relation (2.7) chosen by Einstein for Einstein's equations , corresponding to D?

(2) Can we generalise the model L_{ED} such that the law of conservation to be preserved?

A first step would be that , in a model $L_D = (M, g, D) \overset{g}{\sim} L_\nabla$, to generalise (2.7) namely to consider β an 1-closed form.

(3) Because in the Einstein's equations the curvature tensor $R(XY)Z$ (from which we obtain the Ricci's tensor) plays an important role , it is clear that it might be established a relation between the curvature tensors of the two linear connections from the two g-equivalent models , but also a relation which involve the considered 1-closed form.

§3.Some curvature invariants for the Einstein relativistique ,g-equivalents models

In the general case $L_D = (M, g, D)$ for the Einstein's equations

(3.1) $$\overset{D}{r}(XY) - \frac{\overset{D}{r}}{2} g(XY) = k \overset{D}{T}(XY)$$

(where $\overset{D}{r}(XY)$ is the Ricci's tensor), even we are not considering the conservation law

(3.2) $div_D T = 0$

occur problems from mathematical point of view , but also regarding physical interpretations .

In the general case we have:

Teorema (3.1)([2], [4]). If

(3.3) $$L_{\underset{D}{(2)}} \overset{g}{\sim} L_{\underset{D}{(1)}}$$

then we will have the curvature invariant

(3.4) $$g\left(Y, \overset{(2)}{R}(V,X)Z - \frac{1}{n}\overset{(2)}{B}(V,X)_{\bullet} Z\right) +$$

$$+ g\left(\overset{(1)}{R}(V,X)Y - \frac{1}{n}\overset{(1)}{B}(V,X)_{\bullet} Y, Z\right) = 0$$

$$\forall VXYZ \in \mathfrak{X}(M)$$

where R is the curvature tensor and B is the Bianchi's tensor .

If we are taking account of (1.3) (1.4) we will get :

Theorem (3.2). If $L_{\underset{D}{(1)}} \overset{g}{\sim} L_{\underset{D}{(2)}}$ then we will have the general relation

(3.5) $$\overset{(1)}{B}(X,Y) + \overset{(2)}{B}(X,Y) = n(d\alpha)(Y,X)$$

An extended , purely geometrical theory is given [2], [4].

Let us assume that $\overset{1}{D} = \nabla, \overset{2}{D} = D$ (an arbitrary one) and $L_D \overset{g}{\sim} L_{\nabla}$.

Because $\overset{\nabla}{B} = 0$ it results

(3.6) $\overset{(2)}{B}(X,Y)=n(d\alpha)(YX) \qquad \forall X,Y$

(3.7) $\overset{(2)}{r}(X,Y)=(d\alpha)(X,Y)+\overset{\nabla}{r}(XY)$

Because

(3.8) $\overset{\nabla}{r}(X,Y)=\overset{\nabla}{r}(Y,X)$

it results :

(3.9) $\overset{(2)}{r}(X,Y)-\overset{(2)}{r}(Y,X)=2(d\alpha)(XY)$

We have :

Theorem (3.3). If $L_D \overset{g}{\sim} L_\nabla$ then we will have the relations (3.6) , (3.7), (3.9).

Theorem (3.4). If $L_D \overset{g}{\sim} L_\nabla$ then the Ricci's tensor $\overset{(2)}{r}(XY)$ is a symmetrical one if and only if the 1-form (α) is a closed one :

(3.10) $d\alpha=0$

Theorem (3.5). If $L_D \overset{g}{\sim} L_\nabla$ then we will have

(3.11) $\overset{D}{r}(X,Y)=\overset{\nabla}{r}(XY)$

if and only if (α) is a closed form .

Theorem (3.6). If $L_D \overset{g}{\sim} L_\nabla$ and $d\alpha=0$ then the scalar curvatures are equal $\left(\overset{D}{r}=\overset{\nabla}{r}\right)$.

As a particular case we have:

Theorem (3.7). For every nonsymmetrical Einstein model we have the above results .

So every model $L_D \overset{g}{\sim} L_\nabla$ where α is a closed 1-form is more general than the nonsymmetrical Einstein model.

Let us write the Einstein's equations :

(3.12) $$\overset{\nabla}{r}(XY) - \frac{\overset{\nabla}{r}}{2} g(XY) = \overset{\nabla}{k}\overset{\nabla}{T}(X,Y) \quad \forall X,Y$$

As it is already known we have the conservation law

(3.13) $$div_\nabla \overset{\nabla}{T} = 0$$

and it is verified also in this case.

Now for the equations (3.1) we have a powerful meaning. We have :

(3.14) $$\overset{\nabla}{r}(XY) - \frac{\overset{D}{r}}{2} g(XY) = \overset{\nabla}{r}(XY) - \frac{\overset{D}{r}}{2} g(XY) = \overset{\nabla}{k}\overset{\nabla}{T}(XY)$$

i.e. :

(3.15) $$\overset{D}{k}\overset{D}{T} = \overset{\nabla}{k}\overset{\nabla}{T}$$

It results :

Theorem (3.8). If $L_D \overset{g}{\sim} L_\nabla$ and α is a closed 1-form then the formal equations (3.1) have physical meaning.

Obviously we still have to prove the conservation law related to $\overset{D}{T}$, i.e. we have to prove

(3.16) $$div_D \overset{D}{T} = 0$$

From (3.15) and (2.4) it results :

Theorem (3.9). If $L_D \overset{g}{\sim} L_\nabla$ where α is a closed 1-form then we will have

(3.17) $\overset{D}{k}\, div_D \overset{D}{T} = \overset{1}{k}\, div_\nabla \overset{\nabla}{T}$

From (3.13) and (3.17) it results :

Theorem (3.10). If $L_D \overset{g}{\sim} L_\nabla$ where α is a closed 1-form then the conservation law for L_D is preserved . We have (3.2).

As a particular case when α is an exact 1-form all the above results remain true . Thus we have a generalisation of the nonsymmetrical Einstein ' s model starting from a geometrical aspect (the parallel transport of the distributions).

Starting from the already known idea that , if it is given a vector bundle $\tau_1 = (E, \pi; M)$, where the base M is an (n+m) dimensional, paracompact , connected C^∞-differentiable manifold, we can organise the total space E as a paracompact , C^∞ -differentiable manifold , with the m-dimensional type-fiber ([1]), we can obtain a decomposition $TE = HE \oplus VE$ (Whitney sum) where H is an horizontal distribution and V is a vertical distribution , which are defined by g (on E) (H and V are orthogonal ones). In this case there exists linear connections $\{D\}$ on E which preserve by parallelism these distributions . Using these connections the relation $\overset{g}{\sim}$ will be analysed only for the horizontal 1-dimensional distributions , respectively the vertical 1-dimensional distributions . Applying the above results we can give a meaning of the formal equations (3.1),which appear in the vector bundle theory .

A special case , which is closer to the physical meaning , of Einstein' s type , is that one when the vertical fiber is an 1-dimensional one and the signature of g is a Lorentz signature .

Our research on these cases is in progress.

References

1.[Mir-An]- Miron R., Anastasiei M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

2.[St] – Stavre , P. , Fibrate vectoriale, EUC – 2005 – vol. II, 2007, vol. III

3.[St] –Stavre , P. , Miron-Norden models of Lagrange type Anal. St. Univ. Al. I. Cuza – Iaşi, 2007,pp.317-325

4.[St. L] – Stavre , P. , Lupu , A., „Almost Symplectic Conjugated Structures on the Total Space of a Vector Bundle . Applications "(in Romanian), Ed. MatrixRom, Bucuresti , 2009

5.[St. L] – Stavre , P. , Lupu , A. , „Cosmology , the General Theory of Relativity and g- conjugated models" -27th Texas Symposium on Relativistic Astrophysics , The University of Texas at Dallas ,Dallas , USA , December 8-13 , 2013

http://nsm.utdallas.edu/texas2013/proceedings/1/4/f/Lupu.pdf

6. [Steenrod] – Steenrod N., *Topology of fibre bundle*, Princeton University, 1951

7. [Einstein] – Einstein, A. , "The Meaning of Relativity" - Fifth edition –Princeton University Press- 1955