Equations of Motion for Radiating Black Holes

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Motivation and Background

- Black holes radiate*

$$temperature \sim \frac{1}{mass}$$

- May be relevant to understanding:
  - end state of black holes
  - information loss problem
  - singularity resolution
- Formation of radiating black holes is difficult to model, especially in
  - > 2 Dimensions
  - in non-null co–ordinates, $ds^2 \neq e^{2f}dudv$

*Hawking, Comm Math Phys (1975)

Objectives

Long term goals:

- Perform numerical simulations of radiating black hole formation
- Understand the end point of black hole (BH) formation
- Can the singularity be removed by hand?*

*Ziprick & Kunstatter, Phys Rev D (2009)

Objectives

Current objectives:

1. Set up a "higher dimensional" toy model based on 2D physics
2. Find the equations of motion (EoM) in null co–ordinates
    - Good for numerical simulations - Automatic mesh refinement
3. Find Hamilton's EoM in R-T co–ordinates
    - Simulate past horizon formation
    - 1^{st} order equations (more stable numerically)

Quantum Corrections in 2D

- In 2D we can add a quantum correction to the energy–momentum (EM) tensor

$$< T^Q >= \begin{bmatrix} f_{,uu} - {f_{,u}}^2 & -f_{,uv} \\ -f_{,uv} & f_{,vv} - {f_{,v}}^2 \end{bmatrix}$$

  where $ds^2 = e^{2f} dudv$
- Only valid in 2D
- Only valid for minimally coupled matter
- It has been used in higher dimensional, toy models

Ayal and Piran's* Model:

- Assumed 4D, spherical symmetry, $ds^2 = e^{2f}dudv - R^2 d\Omega^2$
- Found equation of motion from

$$G = -8\pi T + Q(R)\begin{bmatrix} f_{,uu} - {f_{,u}}^2 & -f_{,uv} & 0 & 0 \\ -f_{,uv} & f_{,vv} - {f_{,v}}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - G = Einstein tensor
  - T = EM tensor of a scalar field, ψ
  - $Q = \alpha/4\pi R^2$ is a higher dimensional correction factor
  - α = coupling constant

*Ayal & Piran, Phys Rev D (1997), Parentani & Piran, PRL (1994)

Also see Ashtekar, Pretorius et al ∼2011 for related model

- Numerically solved the equations of motion
- Investigated black hole formation
- Altered $Q = \alpha/4\pi R^2$ to remove singularity

$$Q = \frac{1}{4\pi}\frac{(\alpha/R^2)^2}{1+(\alpha/R^2)^2}$$

- This gives non-divergenceless EM tensor

$$\nabla^\mu(T_{\mu\nu} + Q < T^Q >_{\mu\nu}) \neq 0$$

- Most noticable when $R < \sqrt{\alpha}$

- What about an action formulation?
- The Polyakov action* accounts for HR

$$I_{Poly} \sim -\int d^2x\sqrt{-g}\mathcal{R}\frac{1}{D^2}\mathcal{R}$$

  - g = determinant of the metric, g_{AB}
  - $\mathcal{R}$ = Ricci scalar
  - D = covariant derivative compatible with g_{AB}
- $< T^Q >_{AB} = \frac{2}{\sqrt{-g}}\frac{\delta I_{Poly}}{\delta g^{AB}}$
- Only valid in 2D
- Assumes the matter field is minimally coupled, for example

$$I_{matter} \sim \int d^2x\sqrt{-g}(D\psi)^2$$

*Polyakov, Phys Lett B (1981)

- What about non-minimally coupled matter?
- Scalar field:

$$I_{matter} \sim \int d^2x\sqrt{-g}B(R)(D\psi)^2$$

- Polyakov action is altered

$$I_{Poly} \sim -\int d^2x\sqrt{-g}\left[\mathcal{R}\frac{1}{D^2}\mathcal{R} + b\left(\frac{1}{D^2}\mathcal{R} + \ln\mu^2\right)(DR)^2 + c\mathcal{R}\right]$$

  - b and c are known functions of B*
  - R = area radius
  - μ is a constant (related to renormalization procedure*)

*Medved, PhD thesis (2001)

- How do we deal with $1/D^2$ operator in I_{Poly}?

$$I_{Poly} \sim -\int d^2x\sqrt{-g}\left[\mathcal{R}\frac{1}{D^2}\mathcal{R} + b\left(\frac{1}{D^2}\mathcal{R} + \ln\mu^2\right)(DR)^2 + c\mathcal{R}\right]$$

- In null co–ordinates $\mathcal{R} = 2D^2 f \to \frac{1}{D^2}\mathcal{R} = 2f$
- What about for other co–ordinates?
- Use auxiliary fields z_1, z_2*

$$I_{Poly} \sim -\int \sqrt{-g}[\mathcal{R}(z_1 + z_2) + D_A z_2 D^A z_1 + b(DR)^2(z_1 - \ln\mu^2) + c\mathcal{R}]$$

- The equations of motion (EoM) for z_1 and z_2 are

$$D^2 z_1 - \mathcal{R} = 0, \quad D^2 z_2 - \mathcal{R} - b(DR)^2 = 0$$

- Plugging these in gives the original I_{Poly}

*Hayward, Phys Rev D (1995), Medved, PhD thesis (2001)

Our Model - The Action

- $I = I_{gravity} + I_{Poly} + I_{matter}$
- where
$$I_{gravity} = \frac{1}{l^{n-2}} \int d^2x\sqrt{-g}[\phi(R)\mathcal{R} + h(R)(DR)^2 + V(R)]$$
(Spherically sym GR if $\phi = R^{n-2}$, $V = h = (n-2)(n-3)R^{n-4}$)
-
$$I_{Poly} = \frac{1}{l^{n-2}} \int d^2x\sqrt{-g}\times$$
$$W[\mathcal{R}(z_1 + z_2) + D_A z_2 D^A z_1 b(R)(DR)^2(z_1 - \ln\mu^2) + c(R)\mathcal{R}]$$
-
$$I_{matter} = \frac{1}{l^{n-2}} \int d^2x\sqrt{-g}B(R)(D\psi)^2$$
(Spherically sym scalar field if $B = -8\pi R^{n-2}$)

Null Co–ordinates

- Lagrange's equations: Too long to write out
- Quantum correction to EM tensor on shell is (with $b = c = 0$),

$$< T^Q_{AB} >= Q_{(n)} \begin{bmatrix} f_{,uu} - {f_{,u}}^2 & -f_{,uv} \\ -f_{,uv} & f_{,vv} - {f_{,v}}^2 \end{bmatrix}$$

- where

$$Q_{(n)} \sim \phi^{-1}$$

- ($\phi = R^{n-2}$ in the GR case)
- This is Ayal and Piran's correction

Hamiltonian Analysis in R-T Co–ordinates

- Advantages for simulations:
  - First order in time derivatives - more stable
  - Gauge choice may allow to simulate past horizon formation
- Start with the ADM line element*

$$ds^2 = -N^2 dt^2 + \Lambda^2 (N_r dt + dx)^2$$

where N (lapse), N_r (shift), Λ are functions of x and t

- Useful to define $, y$ operator

$$\beta_{,y} := N^{-1}(\beta_{,t} - N_r \beta_{,x})$$

- All t derivatives show up in this form

*Arnowitt et. al. (1962)

- (Skipping lots of math) The hamiltonian density is

$$\mathcal{H} = NH + N_r H_r$$

where the constraints are

$$\begin{aligned} H =& 2\left(\frac{\tilde{\phi}' R_{,x}}{\Lambda}\right)_{,x} - \frac{h_z {R_{,x}}^2}{\Lambda} - \Lambda V \\ &+ W\left[-\frac{z_{1,x} z_{2,x}}{\Lambda} + 2\left(\frac{z_{1,x} + z_{2,x}}{\Lambda}\right)_{,x}\right] \\ &+ \frac{S_1^2}{4W} - \frac{S_2 {R_{,y}}^2}{4W} - \frac{P_{z1} P_{z2}}{W\Lambda} \\ &- \frac{P_\psi^2}{4\Lambda B} - \frac{B {\psi_{,x}}^2}{\Lambda} \end{aligned}$$

and

$$H_r = -P_{\Lambda,x}\Lambda + P_R R_{,x} + P_{z1} z_{1,x} + P_{z2} z_{2,x} + P_\psi \psi_{,x}$$

Gauge Fixing

- 2 Lagrange multipliers in the Hamiltonian $\rightarrow$ 2 gauge choices: $\chi = R - x = 0$, $\xi = \Lambda - 1 = 0 \rightarrow$

$$g^{AB} = \begin{bmatrix} -N^{-2} & N_r N^2 \\ N_r N^2 & (1 - N_r^2 N^{-2}) \end{bmatrix}$$

  Regular at horizon formation
- N and N_r determined by $\dot{\chi} = \dot{\xi} = 0 \rightarrow$

$$N_r = -\exp\left[\int dR \frac{S_3}{2R_{,y}}\right], \quad N = \frac{1}{R_{,y}} \exp\left[\int dR \frac{S_3}{2R_{,y}}\right]$$

- Must write P_R & P_Λ in terms of ψ, P_ψ, z_1, P_{z1}, z_2, P_{z2} using

$$H = 0, \quad H_r = 0$$

Equations of Motion

- The gauge fixed EoM for the scalar field are

$$\dot{\psi} = -N\frac{P_\psi}{2B} + N_r\psi', \quad \dot{P}_\psi = (-2NB\psi' + N_r P_\psi)'$$

- The gauge fixed EoM for the auxiliary fields are

$$\dot{z}_1 = -NS_3 - \frac{NP_{z2}}{W} + N_r z_1', \quad \dot{P}_{z1} = (-W(Nz_2' + 2N') + N_r P_{z1})' - NWb({R_{,y}}^2 - 1)$$

$$\dot{z}_2 = -NS_3 - \frac{NP_{z1}}{W} + N_r z_2', \quad \dot{P}_{z2} = (-W(Nz_1' + 2N') + N_r P_{z2})'$$

Conclusions

- Set up a toy model for modelling radiation
  - general - many user definable functions
  - allows for non minimally coupled matter
- Found Lagrange's EoM in null co–ordinates
- Found Hamilton's EoM in R–T co–ordinates
  - First order in time derivatives - good for simulations
  - Allow for simulation past horizon formation

Future Work

- Investigate singularity resolution at $R = 0$ by altering ϕ, h, V & B
  - in the static solution (Ziprick & Kunstatter, Phys Rev D (2009))
  - in $Q_{(n)}$ (higher dimensional correction factor in $< T^Q >$)
  - simultaneously?

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